5.1 Combining Functions:

Introduction:

① operations on whole numbers:

$2+3=5$

$2-3=-1$

$2\cdot 3=6$

② operations with fractions/decimals

$2/3+3/4$

$4/5\cdot 5/6$

$4/5\div 5/6$

③ operations with variables

$x+5=6$

$x\cdot 5=5x$

④ operations with functions:

$f(x)+g(x)$

$f(x)\cdot g(x)$

Defn.

Operations on Functions:

If $f(x)$ and $g(x)$ both exist, the sum, difference, product, and quotient of two functions f and g are defined by

$$(f+g)(x) = f(x) + g(x)$$

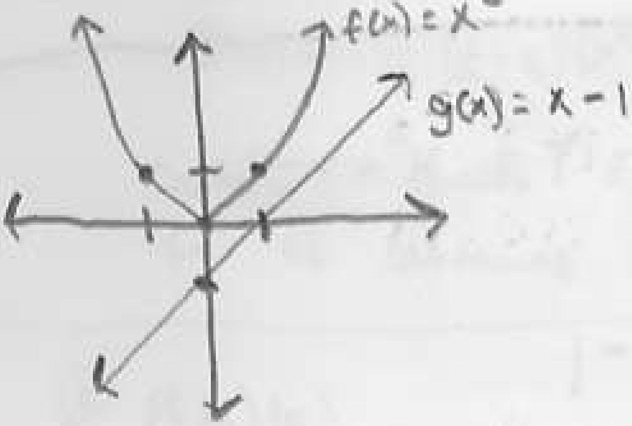
$$(f-g)(x) = f(x) - g(x)$$

$$(fg)(x) = f(x)\,g(x)$$

$$\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)} \quad g(x) \neq 0$$

examples:

graphically:



$(f+g)(0) = f(0) + g(0) = 0 - 1 = -1$ $\quad f(0) = 0$

$(f-g)(0) = f(0) - g(0) = 0 - (-1) = 1$ $\quad g(0) = -1$

$(f \cdot g)(0) = f(0) \cdot g(0) = 0 \cdot (-1) = 0$

$(f/g)(0) = \dfrac{f(0)}{g(0)} = \dfrac{0}{-1} = 0$

Algebraically:

Let $f(x) = x^2$

$g(x) = x - 1$

$(f+g)(x) = f(x) + g(x) = x^2 + (x-1) = x^2 + x - 1$

$(f+g)(x) = x^2 + x - 1$

$(f+g)(0) = 0^2 + 0 - 1 = -1$

$(f-g)(x) = f(x) - g(x) = x^2 - (x-1) = x^2 - x + 1$

$(f-g)(x) = x^2 - x + 1$

$(f-g)(0) = 0^2 - 0 + 1 = 1$

$(f \cdot g)(x) = f(x)g(x) = (x^2)(x-1) = x^3 - x^2$

$(f \cdot g)(x) = x^3 - x^2$

$(f \cdot g)(0) = 0^3 - 0^2 = 0$

$(f/g)(x) = \frac{f(x)}{g(x)} = \frac{x^2}{x-1}$

$(f/g)(x) = \frac{x^2}{x-1}$

$(f/g)(0) = \frac{0^2}{0-1} = 0$

Checking domain:

① $(f+g)(x)$, $(f-g)(x)$, $(f\cdot g)(x)$

Domain includes x-values that are in both in the domains of $f(x)$ and $g(x)$.

② $(f/g)(x)$

Domain includes x-values that are in both in the domains of $f(x)$ and $g(x)$ and $g(x) \neq 0$.

Use $f(x)$ and $g(x)$ to find each expression symbolically. Identify its domain.

(14) $f(x) = 1-4x$ $\quad g(x) = 3x+1$

① Domain $f(x) = \mathbb{R}$ $\quad g(x) = \mathbb{R}$

$(f+g)(x) = (1-4x) + (3x+1) =$

$(f-g)(x) = (1-4x) - (3x+1) =$

$(f\cdot g)(x) = (1-4x)(3x+1) =$

Domain $= \mathbb{R}$

$(f/g)(x) = \dfrac{(1-4x)}{3x+1}$ → Domain Restriction $g(x) \neq 0$ $\boxed{x \neq -1/3}$

$$x + \sqrt{x-1} = 0$$

$$\sqrt{x-1} = -x$$

$$x-1 = x^2$$

$$x^2 - x + 1 = 0$$

$$b = \frac{-1 \pm \sqrt{1-4(1)(1)}}{2}$$

(17) $f(x) = x - \sqrt{x-1}$ $g(x) = x + \sqrt{x-1}$

Domain $f(x) =$

$x - 1 \geq 0$

$x \geq 1$

$\{x \mid x \geq 1\}$

$g(x)$

$x - 1 \geq 0$

$x \geq 1$

$\{x \mid x \geq 1\}$

Domain $(f+g)(x)$, $(f \cdot g)(x)$, $(f-g)(x) = \{x \mid x \geq 1\}$

① $(f+g)(x) = f(x) + g(x) = (x - \sqrt{x-1}) + (x + \sqrt{x-1})$
$= 2x$

② $(f-g)(x) = f(x) - g(x) = (x - \sqrt{x-1}) - (x + \sqrt{x-1})$
$= -2\sqrt{x-1}$

③ $(fg)(x) = f(x) \cdot g(x) = (x - \sqrt{x-1})(x + \sqrt{x-1})$
$= x^2 + x\sqrt{x-1} - x\sqrt{x-1} - (\sqrt{x-1})^2$
$= x^2 - (x-1)$
$= x^2 - x + 1$

④ $(f/g)(x)$

Domain: $\{x \mid x \geq 1\}$ and $g(x) \neq 0$

check:

$x + \sqrt{x-1} = 0$

$\sqrt{x-1} = -x$

$x - 1 = x^2$

$x^2 - x + 1 = 0$

No real solution: so domain stays same

$\{x \mid x \geq 1\}$

$(f/g)(x) = \dfrac{x - \sqrt{x-1}}{x + \sqrt{x-1}}$

(34) $f(x) = \frac{1}{x+2}$ $g(x) = x^2 + x - 2$

Domain

estr: $f(x) = x+2 \neq 0$

$x \neq -2$

Domain $= \{x \mid x \neq -2\}$

$g(x) = x^2 + x - 2$

No restrictions

Domain $= \{x \mid x = \mathbb{R}\}$

Domain for

$(f+g)(x)$, $(f-g)(x)$, $(f \cdot g)(x) = \{x \mid x \neq -2\}$

$(f+g)(x) = f(x) + g(x) = \frac{1}{x+2} + (x^2 + x - 2)$

$-g)(x) = f(x) - g(x) = \frac{1}{x+2} - (x^2 + x - 2)$

$f \cdot g)(x) = f(x)\, g(x) = \left(\frac{1}{x+2}\right)(x^2 + x - 2)$

$= \frac{x^2 + x - 2}{x+2} = \frac{\cancel{(x+2)}(x-1)}{\cancel{(x+2)}} = x-1$

but domain still $= \{x \mid x \neq -2\}$

$(f/g)(x)$

Domain $= \{x \mid x \neq -2\}$ and check

$$x^2+x-2=0$$

$$(x+2)(x-1)=0$$

$$x \neq -2,\ x \neq 1$$

Domain $= \{x \mid x \neq -2,\ x \neq 1\}$

$$(f/g)(x) = \frac{f(x)}{g(x)} = \frac{\frac{1}{x+2}}{(x+2)(x-1)} = \frac{1}{(x+2)^2(x-1)}$$

Always check domain first before simplifying!

Composition of functions:

If f and g are functions, then the composite function $g \circ f$ or composition of g and f is defined by

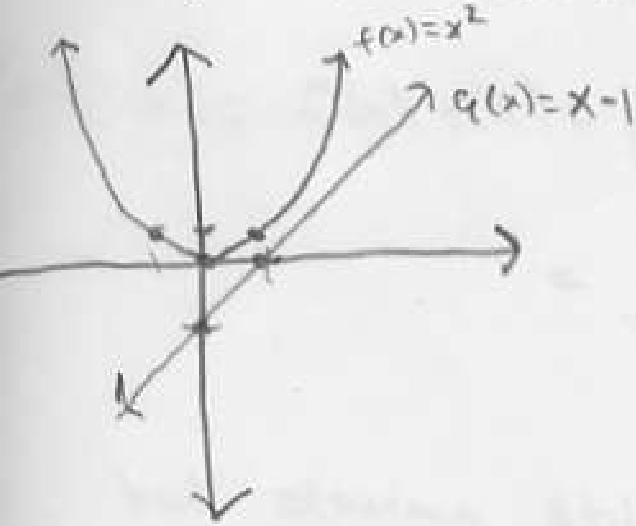
$$(g \circ f)(x) = g(f(x))$$

read $g(f(x))$ = g of f of x.

$$x \to \boxed{f} \to f(x)$$

$$x \to \boxed{f} \to f(x) \to \boxed{g} \to g(f(x))$$

Evaluating composite graphically:



$$(g \circ f)(1) = g(f(1))$$

$\boxed{f(1)=1}$ $\to$ $= g(1)$

$g(1)=0$ $\quad = 0$

$$(f \circ g)(1) = f(g(1))$$

$g(1)=0$ $\quad = f(0)$

$\quad = 0$

IN GENERAL

$(g \circ f)(x)$ does not always equal $(f \circ g)(x)$

Algebraically: $f(x) = x^2$ $\quad g(x) = x - 1$

$(f \circ g)(x) = f(g(x)) = f(x-1) = (x-1)^2$

$$\boxed{(f \circ g)(x) = (x-1)^2}$$

$(f \circ g)(1) = (1-1)^2 = 0$

$(g \circ f)(x) = g(f(x)) = g(x^2) = x^2 - 1$

$$\boxed{(g \circ f)(x) = x^2 - 1}$$

$(g \circ f)(1) = 1^2 - 1 = 0$

Domain: $(f \circ g)(x)$

Domain includes x values that ar both in the domain of $f(x)$ and $g(x)$ and also must be in domain of $(f \circ g)(x)$

★ need to check domain After simplification.

(66) use the given f(x) and g(x) to find each of the following Identify the domain.

$f(x) = 2 - x$ $g(x) = \frac{1}{x^2}$

Domain

$\boxed{f(x) = \mathbb{R}}$ $\boxed{g(x) = \{x \mid x \neq 0\}}$

$(f \circ g)(x) = f(g(x)) = f\left(\frac{1}{x^2}\right) = 2 - \frac{1}{x^2}$

↑ check domain again $\mathbb{R}$

Dom = $\{x \mid x \neq 0\}$

$(g \circ f)(x) = g(f(x)) = g(2-x) = \boxed{\frac{1}{(2-x)^2}}$ ← check domain again

↑ $\boxed{x \neq 2}$

Dom = $\{x \mid x \neq 0, 2\}$