5.3 Exponential functions and models:

Defn:

An exponential function with base a is defined for all real numbers x by $f(x) = a^x$

where $a > 0$ and $a = 1$.

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ex.

$f(x) = 2^x$, $g(x) = \pi^x$ $h(x) = e^x$

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evaluating an exponential function.

Let $f(x) = 3^x$

$f(2) = 3^2 = 9$

$f(3) = 3^3 = 27$

$f(0) = 3^0 = 1$

$f(-1) = 3^{-1} = \frac{1}{3}$

$f(-2) = 3^{-2} = \frac{1}{3^2} = \frac{1}{9}$

Also,

$f(1/2) = 3^{1/2} = \sqrt{3}$

$f(3/2) = 3^{3/2} = \sqrt[2]{3^3}$

$f(\pi) = 3^{3.14...} =$

Must use a calculator to solve.

syntax:

3^(3.14) =

operations with exponential functions:

Review

examples

rule does not apply

$x^0 = 1$ | ex | $\pi^0 = 1$

$x^1 = x$ | $e^1 = e$

$x^{-m} = \frac{1}{x^m}$ | $3^{-2} = \frac{1}{3^2} = \frac{1}{9}$

$(x^m)(x^n) = x^{m+n}$ | $(3^2)(3^3) = 3^5$ | $(2^2)(3^2) = (4)(9) = 36$

$\left(\frac{x^m}{x^n}\right) = x^{m-n}$ | $\frac{3^3}{3^5} = 3^{(3-5)} = 3^{-2}$ | $\frac{2^2}{3^2} = \frac{4}{9}$

$(x^m)^n = x^{mn}$ | $(3^2)^5 = 3^{10}$

$x^{m/n} = \sqrt[n]{x^m}$ | $3^{5/3} = \sqrt[3]{3^5}$

$= \left(\sqrt[3]{3}\right)^5$

Examples :

HW (1-16) Simplify the expression without a calculator.

② $(-3)^{-2} = \frac{1}{(-3)^2} = \frac{1}{9}$

Also

$-(3)^{-2} = \frac{-1}{(3)^2} = -\frac{1}{9}$

⑥ $(-4)(8)^{-2/3} = (-4)\left(\frac{1}{8^{2/3}}\right)$

$= (-4)\left(\frac{1}{(\sqrt[3]{8})^2}\right)$

$= (-4)\left(\frac{1}{2^2}\right)$

$= -4\left(\frac{1}{4}\right)$

$= -1$

⑧ $a^{0.5}\,a^{1.5} = a^{(0.5+1.5)} = a^2$

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⑩ $\dfrac{9^{5/6}}{9^{1/3}} = \dfrac{(3^2)^{5/6}}{(3^2)^{1/3}} = \dfrac{3^{10/6}}{3^{2/3}} = \dfrac{3^{5/3}}{3^{2/3}}$

$= 3^{5/3-2/3} = 3^{3/3} = 3$

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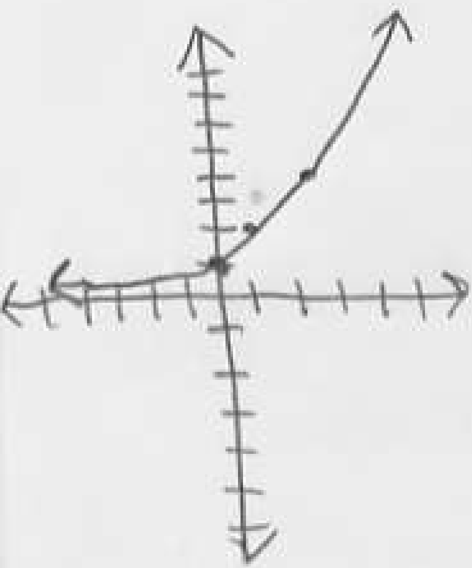
important to remember = there are many ways to solve these problems Any approach can be correct as long as rules are correctly followed.

Graphs of exponential functions:

consider $f(x) = 2^x$

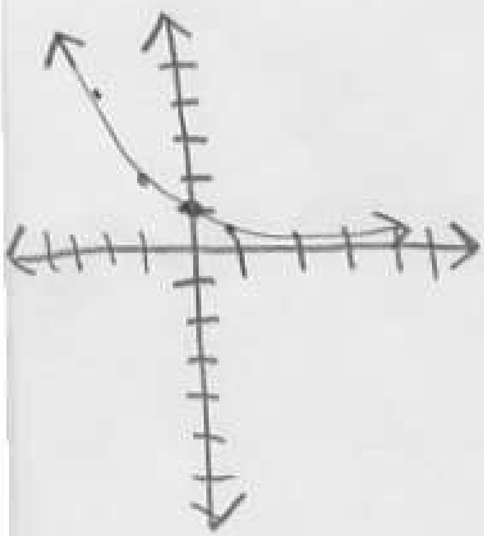


x	$f(x)$	
0	1	2^0
1	2	2^1
2	4	2^2
-1	$1/2$	2^{-1}
-2	$1/4$	2^{-2}

Domain $= \{x \mid x \in R\}$ RANGE $= \{y \mid y > 0\}$

consider $g(x) = 2^{-x}$

$\boxed{g(x) = f(-x)}$



x	$g(x)$	
0	1	
1	$1/2$	2^{-1}
2	$1/4$	2^{-2}
-1	2	2^1
-2	4	2^2

recall Transformations: for $f(x)$.

$-f(x) \Rightarrow$ rotate $f(x)$ around x-axis

$f(-x) \Rightarrow$ rotate $f(x)$ around y-axis

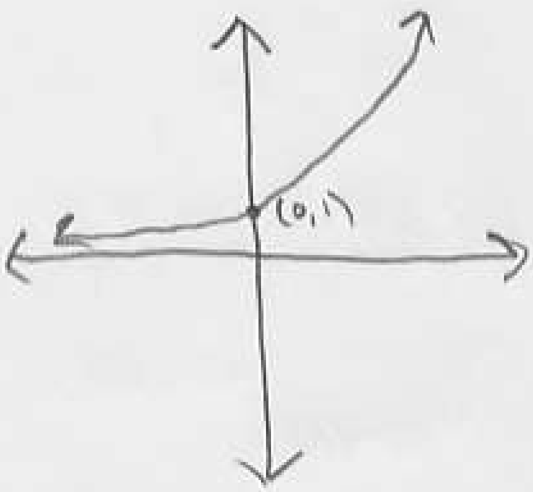
$$g(x)=2^{-x}=(2^{-1})^x=\left(\frac{1}{2}\right)^x$$

Graphs of exponential functions:

The exponential function

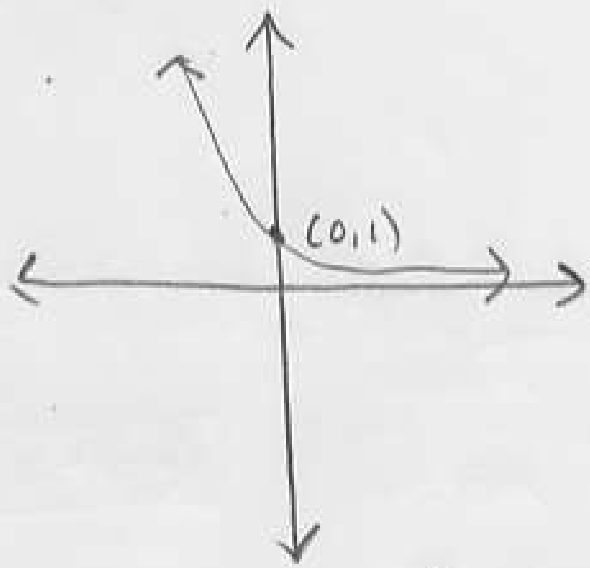
$$f(x)=a^x \qquad \{a>0,\ a\neq 1\}$$

has domain $\mathbb{R}$ and range $(0,\infty)$. The graph has one of these shapes.



$f(x)=a^x \quad a>1$

increasing function

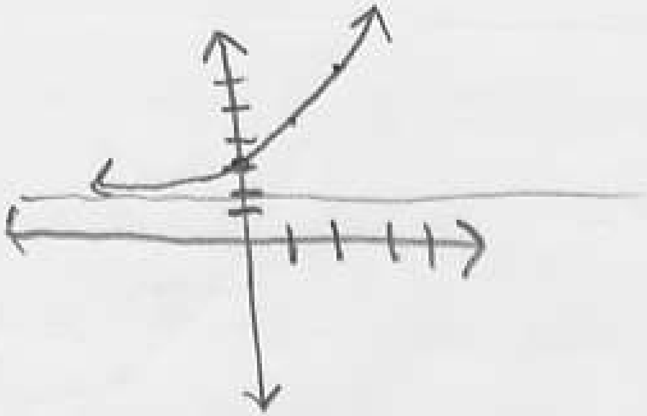


$f(x)=a^x \quad 0<a<1$

decreasing function.

other transformations

$f(x) = 2^x + 2$

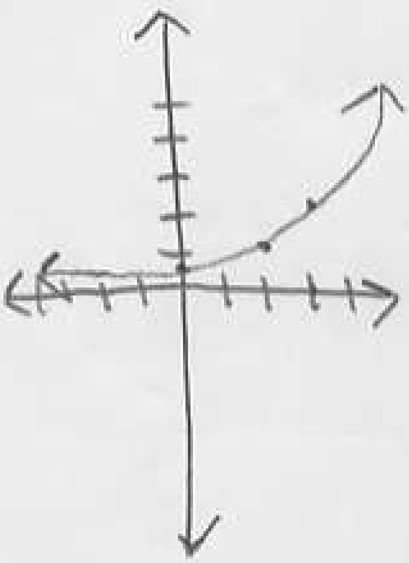


Shift $g(x) = 2^x$ up two units.

x	$f(x)$	
0	3	$2^0 + 2 = 3$
1	4	$2^1 + 2 = 4$
2	6	$2^2 + 2 = 6$
-1		$1/2 + 2 = 2\,1/2$
-2		$= 2\,1/4$

$f(x) = 2^{(x-2)}$

shift $g(x) = 2^x$ right two units



x	$f(x)$	
0	1/4	$= 2^{-2}$
2	1	$= 2^0$
3	2	$= 2^1$
4	4	$= 2^2$
5		
-1		

Exercises: 29-36. Find C and a so that $f(x) = Ca^x$ satisfies the given conditions:

(32)

$f(0)=7$

$f(-1)=1$

$f(0)=7 \Rightarrow 7 = Ca^0$

$\Rightarrow 7 = C(1)$

$\Rightarrow \boxed{7=C}$

$f(-1)=1 \Rightarrow 1 = Ca^{-1}$ but $C=7$

$\Rightarrow 1 = 7a^{-1}$

$\Rightarrow \frac{1}{7} = a^{-1}$

$\Rightarrow \frac{1}{7} = \frac{1}{a}$

$\Rightarrow \boxed{a=7}$

so $f(x) = (7)(7)^x$

$f(x) = 7^{x+1}$

(34)

$f(-1) = \frac{1}{4}$

$f(1) = 4$

$f(x) = Ca^x$

$f(-1) = \frac{1}{4}$

$\Rightarrow \boxed{\frac{1}{4} = Ca^{-1}}$ (1)

$f(1) = 4$

$\Rightarrow \boxed{4 = Ca}$ (2)

Solve eq (2) for C

$C = \frac{4}{a} \Rightarrow C = 4(a^{-1})$

From this

$C = \frac{4}{4} = 1$

Substitute into (1)

$\frac{1}{4} = \underbrace{[(4)(a^{-1})]}_{C}(a^{-1}) \Rightarrow \frac{1}{4} = 4(a^{-2})$

$\frac{1}{16} = \frac{1}{a^2}$

$a = \pm 4 \Rightarrow \boxed{a = 4}$

So $f(x) = 1(4)^x$

$= 4^x$

The natural exponential function is the exponential function

$$f(x) = e^x$$

with base e. It is often referred to as "the exponential function".

<u>Applications:</u> [will do in 5.6]

<u>exponential growth model:</u>

$$n(t) = n_0 e^{rt}$$

$n(t)$ = population at time t.

n_0 = initial population

r = relative rate of growth

t = time.

ex.

A certain breed of rabbit was introduced onto a small island about 8 years ago. The current rabbit population is estimated to be 4100 with a growth rate of 55% per year.

a) what was the initial population of the rabbit population.

b) Estimate the population 12 years from now

$$n(t) = n_0 e^{rt}$$

We know
$r = .55$ and when $t = 8$ $n(t) = 4100$.

$$n(t) = n_0 e^{.55(t)}$$

$$n(8) = 4100 = n_0 e^{(.55)(8)}$$

$$n_0 = \frac{4100}{e^{(.55)(8)}}$$

$$n_0 \approx 50$$

Compound interest:

Derivation:

Assume some P_0 is invested at an interest rate i per time period. The amount of money at time time t, $A(t)$ can be derived the following way.

$$A(0) = P_0$$

$$A(1) = P_0 + i(P_0) = P_0(1+i)$$

$$\begin{aligned} A(2) &= A(1) + i(A(1)) \\ &= P_0(1+i) + i(P_0(1+i)) \\ &= [P_0(1+i)](1+i) \\ &= P_0(1+i)^2 \end{aligned}$$

$$\begin{aligned} A(3) &= A(2) + i(A(2)) \\ &= [P_0(1+i)^2] + i[P_0(1+i)^2] \\ &= P_0(1+i)^2(1+i) \\ &= P_0(1+i)^3 \end{aligned}$$

$$\boxed{A(t) = P_0(1+i)^t}$$

<u>example:</u> A sum of \$1000 is invested at an interest rate of 12% per year. Find the amounts in the account after 3 years. if interest is compounded annually, semi-annually, quarterly, monthly, daily.

$A(t) = P\left(1+\frac{r}{n}\right)^{nt}$ P = 1000 r = .12 t = 3 years

compounding	n	equation	Solution
Annually	1	$=1000\left(1+\frac{.12}{1}\right)^{3}$	\$1404.93
Semi-Annually	2	$=1000\left(1+\frac{.12}{2}\right)^{2(3)}$	\$1418.52
Quarterly	4	$=1000\left(1+\frac{.12}{4}\right)^{(4)(3)}$	\$1425.76
Monthly	12	$=1000\left(1+\frac{.12}{12}\right)^{(12)(3)}$	= \$1430.77
daily	365	$=1000\left(1+\frac{.12}{365}\right)^{3(365)}$	= \$1433.24

recall

$$A(t) = p\left(1+\frac{r}{n}\right)^{nt}$$

what happens if n increases indefinately

So you would have continously compounded interest.

$$A(t) = p\left(1+\frac{r}{n}\right)^{nt}$$

Let $m = n/r \Rightarrow n = rm$

$\Rightarrow \frac{r}{n} = \frac{1}{m}$

$nt = rmt$

$$= p\left(1+\frac{1}{m}\right)^{rmt}$$

$$= p\left[\left(1+\frac{1}{m}\right)^{m}\right]^{rt}$$

Now if $N \to \infty$ $\quad m = n/r \Rightarrow \infty$

what does

$$\left(1+\frac{1}{m}\right)^{m} = \text{ when } m \Rightarrow \infty = e$$

$$\boxed{A(t) = pe^{rt}}$$

so for continously compounded interest

$$A(t) = pe^{rt}$$

1000% dollars invested at 12% compound continously for three years

=

$$\boxed{\begin{aligned} A(3) &= 1000(e^{.12(3)}) \\ &= \$1433.33 \end{aligned}}$$