5.6 Exponential and logarithmic equations

<u>Exponential equations:</u> An equation in which the variable occurs in the exponent.

$2^x = 5$ recall $\ln a^x = x \ln a$

$\ln 2^x = \ln 5$ ← take ln of both sides

$x \ln 2 = \ln 5$ ← solve for x.

$x = \frac{\ln 5}{\ln 2}$ ← use calculator

Guidelines for solving exponential equations:

(1) Isolate the exponential expression on one side of the equation

(2) Take the logarithm of each side, then use the laws of logarithms to bring down exponent

(3) Solve for the variable.

⑥ $2e^{-x} = 8$

$e^{-x} = 4$

$\ln e^{-x} = \ln 4$

$-x \ln e = \ln 4$

$-x = \ln 4$

$x = -\ln 4 \approx -1.386$

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⑫ $0.05(1.15)^x = 5$

$(1.15)^x = \frac{5}{.05}$

$1.15^x = 100$

$\ln(1.15^x) = \ln(100)$

$x\ln(1.15) = \ln(100)$

$x = \frac{\ln(100)}{\ln(1.15)} \approx 32.95$

㉔ $3^{1-2x} = e^{0.5x}$

$\ln 3^{1-2x} = \ln e^{0.5x}$

$(1-2x)\ln 3 = 0.5x \ln e$

$\ln 3 - (\ln 3)2x = 0.5x$

$\ln 3 = \ln(3)(2x) + 0.5x$

$\ln 3 = x(2\ln 3 + .05)$

$x = \frac{\ln 3}{2\ln 3 + .05}$

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$2^x = -4$

$\ln(2^x) = \ln(-4)$ ← No real solution

So no real solution

logarithmic equations: an equation that has a logarithm of a variable.

Guidelines for solving logarithmic equations:

① isolate the logarithmic term on one side of the equation, you may first need to combine logarithmic terms

② write equation in exponential form

③ solve for the variable

examples:

ⓐ $\ln x = 8$

$\log_e x = 8$

$x = e^8 \approx 2981$

ⓑ $\log_2(25 - x) = 3$

$25 - x = 2^3$

$25 - x = 8$

$-x = -17$

$x = 17$

(44) $100 + 10\log x = 120$

$10\log x = -410$

$\log x = -4$

$\boxed{10^{-4} = x}$

(50) $\ln 2x + \ln 3x = \ln 6$

$\ln[(2x)(3x)] - \ln 6 = 0$

$\ln\left[\frac{6x^2}{6}\right] = 0$

$\ln[x^2] = 0$

$e^0 = x^2$

$\boxed{x = \pm 1}$

$\boxed{\text{but } x \neq -1}$

(56) $\ln(x^2-4) - \ln(x+2) = \ln(3-x)$

$\ln\left(\frac{x^2-4}{x+2}\right) - \ln(3-x) = 0$

$\ln(x-2) - \ln(3-x) = 0$

$\ln\left(\frac{x-2}{3-x}\right) = 0$

$e^0 = \frac{x-2}{3-x}$

$3-x = x-2$

$5 = 2x$

$\boxed{5/2 = x}$

Applications of logarithmic and exponential equations:

① continuous compound interest:

ⓐ $A(t) = P\left(1+\frac{r}{n}\right)^{nt}$ ⇒ Interest compounded n times per year

ⓑ $\boxed{A(t) = Pe^{rt}}$ ⇒ Interest compounded continously

Example:

A sum of \$1000 is invested at an interest rate of 8% per year Find the amount of time for the amount to grow to \$4000.

$P = \$1000$

$A(t) = \$4000$

$r = .08$

$t = ?$

$$4000 = 1000e^{.08t}$$

$$\Rightarrow 4 = e^{.08t}$$

$$\ln 4 = \ln e^{.08t}$$

$$\ln 4 = .08t$$

$$\frac{\ln 4}{.08} = t$$

$$\boxed{17.33 \text{ years} \approx t}$$

② Exponential Growth Model - Population.

A population that experiences <u>exponential growth</u> increases according to the model

$$n(t) = n_0 e^{rt}$$

$n(t)$ = population at time t

n_0 = initial population

r = relative rate of growth (expressed as proportion of population)

t = time.

The population of the world was 6.1 billion in 2000 and the estimate relative growth rate was 1.4% per year. Write a function to describe the population growth

ⓐ

$n(t) = ?$

$t = ?$ $\Rightarrow$ $n(t) = 6.1\, e^{(.014)(t)}$

$n_0 = 6.1$ billion

$e =$

$r = .014$

Assuming this model, when will the population reach 122 billion?

$$122 = 6.1\, e^{.014t}$$

$$\ln\left(\frac{122}{6.1}\right) = .014t \Rightarrow t = \frac{\ln\left(\frac{122}{6.1}\right)}{.014} \approx \underline{213.98 \text{ years}}$$

(b) Another way to set up problem

$$n(t) = n_0 (1+r)^{t-t_0}$$

r = growth rate

n_0 = initial population

t = time

t_0 = initial year

n(t) = number at time t.

$$n(t) = 6.1 (1.014)^{t-2000}$$

$$\frac{122}{6.1} = (1.014)^{t-200}$$

$$\ln(20) = (t-2000) \ln(1.014)$$

$$\frac{\ln(20)}{\ln(1.014)} = t - 2000$$

$$t = 2000 + \frac{\ln(20)}{\ln(1.014)} \approx 2215$$

③ radioactive decay

If m_0 is the initial mass of a radioactive substance with half-life h then the mass remaining at time t is modeled by the function:

$$m(t) = m_0 e^{-rt}$$

where $r = \frac{\ln 2}{h}$ $\quad$ $m(t)$ = mass at time t

m_0 = initial mass

A wooden artifact from an ancient tomb contains 65% of the carbon-14 that is present in living trees. How long ago was the artifact made

half-life of carbon 14 is 5730 years

$m_0 = m_0$

$m(t) = .65(m_0)$

$h = 5730$

$r = \frac{\ln 2}{5730} = .00012$

$$\Rightarrow .65 m_0 = m_0 e^{-.00012t}$$

$$.65 = e^{-.00012t}$$

$$\ln(.65) = -.00012t$$

$$t = \frac{\ln(.65)}{-.00012} \approx 3590 \text{ years}$$

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The percentage of radioactive carbon-14 remaining in a fossil after t years is given by $P = 100\left(\frac{1}{2}\right)^{t/5700}$. Suppose a fossil contains 45% of the carbon 14 it contained when it was alive. Estimate the age of fossil.

$$45 = 100\left(\tfrac{1}{2}\right)^{t/5700}$$

$$\frac{45}{100} = \left(\tfrac{1}{2}\right)^{t/5700}$$

$$\frac{9}{20} = \left(\tfrac{1}{2}\right)^{t/5700}$$

$$\ln\left(\tfrac{9}{20}\right) = \left(t/5700\right)\ln\left(\tfrac{1}{2}\right)$$

$$\frac{\ln\left(9/20\right)}{\ln\left(1/2\right)} = t/5700$$

$$5700\left(\frac{\ln(9/20)}{\ln(1/2)}\right) = t \qquad t \approx 6566 \text{ years}$$

Newtons Law of cooling:

If D_0 is the initial temperature difference between an object and its surroundings, and if its surroundings have temperature T_s, then the temperature of the object at time t is modeled by the function

$$T(t) = T_s + D_0 e^{-kt}$$

k = positive constant that depends on insulation of object

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(23) rev. A pan of boiling water with temperature of 212° F is set in a bin of ice with temp of 32° F. The pan cools to 70° F in 60 minutes

(a) Find the formula for $T(t)$

$T_s = 32°$

$D_0 = 212 - 32 = 180°$

$k = ?$

$T(t) = 32 + 180 e^{-k60}$

$70 = 32 + 180 e^{-k60}$

$37 = 180 e^{-kt}$

$\frac{37}{180} = e^{-k60}$

$\ln\left(\frac{37}{180}\right) = -k60$

$k = \frac{-\ln\left(\frac{37}{180}\right)}{60}$

$k = .027$

$T(t) = 32 + 180\,e^{-.027(t)}$

(b) Find temperature of pan after ten minutes

$T(10) = 32 + 180\,e^{-(.027)(10)}$

(c) How long does it take pan to reach 40 degrees?

$40 = 32 + 180\,e^{-(.027)(t)}$

Solve for $\underline{t}$

Logarithmic Scales:

ph scale

$$ph = -\log[H^+]$$

where H^+ is the concentration of hydrogen ions measured in moles/liter

neutral

$\longleftarrow$ 7 $\longrightarrow$

acidic ph basic

(a) The hydrogen concentration of a sample of human blood was measured to be $H^+ = 3.16 \times 10^{-8}$. Find the ph and classify the blood as acidic or basic.

$$ph = -\log[H^+] = -\log[3.16 \times 10^{-8}] \approx 7.5$$

basic